

Weber-Schafheitlin integrals with arbitrary exponent

Michał Wrochna

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Abstract

We present explicit formulae for Weber-Schafheitlin type integrals and give them an interpretation as the kernel of a physically relevant operator related to the hamiltonian of Aharonov and Bohm. In particular, we derive explicit formulae for Weber-Schafheitlin type integrals with exponent larger or equal 1, which are distributions on \mathbb{R}_+ . We discuss several special cases.

1 Introduction

Our aim is to calculate the integral

$$\int_0^\infty \kappa^\rho J_\mu(x\kappa) J_\nu(\kappa) d\kappa, \quad (1.1)$$

for suitable $\mu, \nu \in \mathbb{C}$ and $x \in \mathbb{R}_+ := (0, \infty)$, where J_μ is the Bessel function of the first kind of order μ . In the literature, it is known as the Weber-Schafheitlin discontinuous integral with exponent ρ (or $-\rho$, depending on the convention).

In the case when $\operatorname{Re} \rho < 1$ (and under some additional assumptions on μ, ν) it is convergent to a function, in general not continuous in $x = 1$. It has been derived in several ways and analysed in many special cases, for which we refer in particular to [W] and [DF]. It has been applied in numerous problems, let us mention here only two recent works — [HN], [SS].

The case $\operatorname{Re} \rho \geq 1$ is more delicate and requires a distributional approach. Nevertheless, it is quite natural to consider it; indeed, it appears in some problems where it plays an important role ([KR], [KeR]). In addition to that, we show in this paper that it is the kernel of an operator physically relevant for the Aharonov-Bohm system. There have already been successful attempts to derive useful expressions for the distributional case of (1.1) for special values of parameters, by Kellendonk and Richard [KR] (for $\rho = 1$), by Miroshin [M] ($\rho \rightarrow 1$ asymptotic) and by Salamon and Walter [SW] (recurrence formulae and special values of parameters with $\rho \in \mathbb{Z}$). Motivated by the wish of exhausting all unsolved cases, we provide explicit formulae for (1.1) for arbitrary ρ with positive real part. We discuss also some special cases and compare them with the results mentioned earlier.

The paper is constructed as follows. Section 2 serves as an additional motivation, linking the hamiltonian of Aharonov and Bohm to the integral we discuss further. In Section 3, we derive formulae for integrals involving the modified Bessel function of the second type K_μ , as in the approach of Dixon and Ferrar [DF].

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We then use them to compute the integrals involving the Hankel functions of the first and second kind — H_μ^+ , H_μ^- ,

$$\int_0^\infty \kappa^\rho H_\mu^\pm(x\kappa) J_\nu(\kappa) d\kappa, \quad (1.2)$$

closely related to (1.1). Our treatment of the integrals (1.2) is a generalization of the approach adopted in [KR]. We consider separately the cases $\operatorname{Re} \rho \leq 0$ and $\operatorname{Re} \rho > 0$. The second one is examined in Section 4 and includes in particular the distributional cases. The main result for the integral (1.1) is contained in Proposition 1.1, and follows as a simple consequence of the computations of (1.2).

1.1 The main result

In the following proposition, we give formulae for the Weber-Schaftheitlin integral (1.1) with exponent $\operatorname{Re} \rho > 0$. The result is a distribution on \mathbb{R}_+ (in general not regular), and its form depends on whether ρ is an integer number (which is not the case if $\operatorname{Re} \rho \leq 0$). We use the notation for the rescaled Gauss hypergeometric function

$${}_2F_1^I(a, b; c; z) := \frac{1}{\Gamma(c+1)} {}_2F_1(a, b; c; z).$$

Proposition 1.1 *For any $\mu, \nu \in \mathbb{C}$ and $\operatorname{Re} \rho > 0$ satisfying $\operatorname{Re}(\rho + \nu + 1) > |\mu|$, and $x \in \mathbb{R}_+$, the integral $\int_0^\infty \kappa^\rho J_\mu(x\kappa) J_\nu(\kappa) d\kappa$ (1.1) equals:*

- for $\rho \notin \mathbb{Z}$,

$$\begin{aligned} & \frac{2^\rho}{\sin \pi \rho} \left[\left\{ (x-1)_-^{-\rho} \sin\left(\pi \frac{1-\rho-\mu+\nu}{2}\right) + (x-1)_+^{-\rho} \sin\left(\pi \frac{1+\rho-\mu+\nu}{2}\right) \right\} \frac{x^{-1+\rho-\nu}}{(1+x)^\rho} {}_2F_1^I\left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; -\rho+1; 1-x^{-2}\right) \right. \\ & \quad \left. - \sin\left(\pi \frac{1+\rho-\mu+\nu}{2}\right) \frac{\Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\rho-\mu+\nu}{2}\right)}{\Gamma\left(\frac{1-\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\rho-\mu+\nu}{2}\right)} x^{-1-\rho-\nu} {}_2F_1^I\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \rho+1; 1-x^{-2}\right) \right] \end{aligned}$$

- for $\rho \in \mathbb{Z}$ and $\frac{1-\rho\pm\mu+\nu}{2} \notin \mathbb{Z}$,

$$\begin{aligned} & \frac{2^\rho}{\pi} \left[\left\{ \sin\left(\pi \frac{1+\rho-\mu+\nu}{2}\right) \left[(x-1)_-^{-\rho} + (-1)^\rho (x-1)_+^{-\rho} \right] + \cos\left(\pi \frac{1+\rho+\mu-\nu}{2}\right) (-1)^\rho \pi \frac{\delta^{(\rho-1)}(1-x)}{(\rho-1)!} \right\} \right. \\ & \quad \times x^{-1+\rho-\nu} (1+x)^{-\rho} S_{\mu,\nu,\rho}(1-x^{-2}) + (-1)^\rho \frac{\Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\rho-\mu+\nu}{2}\right)}{\Gamma\left(\frac{1-\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\rho-\mu+\nu}{2}\right)} x^{-1-\rho-\nu} \left\{ \sin\left(\pi \frac{1+\rho-\mu+\nu}{2}\right) T_{\mu,\nu,\rho}(1-x^{-2}) \right. \\ & \quad \left. \left. - \left[\sin\left(\pi \frac{1+\rho-\mu+\nu}{2}\right) \log x^{-2}(x+1)|x-1| + \cos\left(\pi \frac{1+\rho-\mu+\nu}{2}\right) \pi \theta(x-1) \right] {}_2F_1^I\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \rho+1; 1-x^{-2}\right) \right\} \right]. \end{aligned}$$

where all the equalities hold in the sense of distributions on \mathbb{R}_+ . In both cases, the RHS is the sum of a distribution multiplied by a smooth function (the first term) and of a locally integrable function (the second term). We have denoted $\theta(x)$ the Heaviside theta function, and the functions $S_{\mu,\nu,\rho}, T_{\mu,\nu,\rho}$ are defined for $|z| < 1$ by

$$S_{\mu,\nu,\rho}(z) := \sum_{k=0}^{\rho-1} \frac{\left(\frac{1-\rho+\mu+\nu}{2}\right)_k \left(\frac{1-\rho-\mu+\nu}{2}\right)_k}{(1-\rho)_k k!} z^k,$$

$$\begin{aligned} & T_{\mu,\nu,\rho}(z) := \sum_{k=0}^{\infty} \frac{\left(\frac{1+\rho+\mu+\nu}{2}\right)_k \left(\frac{1+\rho-\mu+\nu}{2}\right)_k}{(\rho+k)! k!} z^k \\ & \times \left\{ \psi(k+1) + \psi(\rho+k+1) - \psi\left(\frac{1+\rho+\mu+\nu}{2} + k\right) - \psi\left(\frac{1+\rho-\mu+\nu}{2} + k\right) \right\}, \end{aligned}$$

with $\psi(y) := \frac{d}{dy}\Gamma(y)/\Gamma(y)$. We use also the definition, for $\operatorname{Re} \lambda > -1$:

$$(x-1)_-^\lambda := \begin{cases} |x-1|^\lambda, & x < 1 \\ 0, & x \geq 1 \end{cases},$$

$$(x-1)_+^\lambda := \begin{cases} 0, & x < 1 \\ (x-1)^\lambda, & x \geq 1 \end{cases},$$

and extend it analytically in the sense of distributions to all values of $\lambda \in \mathbb{C}$ (see Remark 4.2).

2 The Weber-Schafheitlin integral as an integral kernel

In the following section, we quote some results obtained in [BDG] for the Aharonov-Bohm hamiltonian and conclude from them, that the Weber-Schafheitlin integral describes the integral kernel of a physically relevant operator. We motivate thus the need for explicit formulae, valid in particular for the distributional $\operatorname{Re} \rho \geq 1$ case.

We consider the Hilbert space $L^2(\mathbb{R}^2)$ and denote its inner product $(\cdot|\cdot)$. Since it will be convenient to use polar coordinates r, ϕ on \mathbb{R}^2 , we introduce the unitary transformation

$$L^2(\mathbb{R}^2) \ni f \mapsto Uf \in L^2(0, \infty) \otimes L^2(-\pi, \pi)$$

given by $Uf(r, \phi) = \sqrt{r}f(r \cos \phi, r \sin \phi)$, which allows us to identify $L^2(\mathbb{R}^2)$ with $L^2(0, \infty) \otimes L^2(-\pi, \pi)$.

The Aharonov-Bohm hamiltonian in polar coordinates is

$$H_\lambda^{\text{AB}} := -\partial_r^2 - \frac{1}{r^2}(\partial_\phi + i\lambda)^2,$$

understood as the self-adjoint operator associated to the differential expression above (defined on an appropriate domain). We allow for the moment the parameter λ to be any complex number.

Since the self-adjoint operator $L := -i\partial_\phi$ has spectrum $\operatorname{sp}(L) = \mathbb{Z}$, we have the decomposition $L^2(\mathbb{R}^2) = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k$ where \mathcal{H}_k is the spectral subspace of L for the eigenvalue k . With the help of U we can identify \mathcal{H}_k with $L^2(\mathbb{R})$. Since L commutes with H_λ^{AB} , we obtain the decomposition

$$UH_\lambda^{\text{AB}}U^* = \bigoplus_{k \in \mathbb{Z}} H_{k+\lambda},$$

where H_μ acts as the differential operator $-\partial_x^2 + \frac{\mu^2 - \frac{1}{4}}{x^2}$, when restricted to $\mathcal{C}_c(\mathbb{R}_+)$.

We now assume $\mu > -1$. We gather some results from [BDG] about the operator H_μ . We will need first to define the following symmetric operator, corresponding up to a constant factor to the so-called Hankel transformation:

Definition 2.1 \mathcal{F}_μ is the operator on $L^2(0, \infty)$ given by

$$(\mathcal{F}_\mu f)(k) := \int_0^\infty J_\mu(kx) \sqrt{kx} f(x) dx$$

We have then:

Theorem 2.2 Let $0 < a < b < \infty$ and denote $\mathbb{1}_{[a,b]}$ the characteristic function of the corresponding interval. The integral kernel of $\mathbb{1}_{[a,b]}(H_\mu)$ is

$$\mathbb{1}_{[a,b]}(H_\mu)(x, y) = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_\mu(x\kappa) J_\mu(y\kappa) \kappa d\kappa,$$

considered as a quadratic form on $\mathcal{C}_c^\infty(\mathbb{R}_+)$, that is, explicitly:

$$(f | \mathbb{1}_{[a,b]}(H_\mu) f) = \int_0^\infty \mathbb{1}_{[a,b]}(\kappa^2) |(\mathcal{F}_\mu f)(\kappa)|^2 d\kappa$$

for any $f \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. We may thus identify

$$\mathbb{1}_{[a,b]}(H_\mu) = \mathcal{F}_\mu \mathbb{1}_{[a,b]}(Q^2) \mathcal{F}_\mu^*$$

where Q is the self-adjoint position operator, and in consequence,

$$\mathcal{F}_\mu H_\mu \mathcal{F}_\mu^{-1} = Q^2.$$

Note that in particular, \mathcal{F}_μ is a unitary involution. For any $\gamma \in \mathbb{C}$, one gets

$$\mathcal{F}_\mu H_\mu^\gamma \mathcal{F}_\mu^{-1} = Q^{2\gamma},$$

and in the sense above, the integral kernel of H_μ^γ is

$$H_\mu^\gamma(x, y) = \int_0^\infty \kappa^{2\gamma} \sqrt{xy} J_\mu(x\kappa) J_\mu(y\kappa) \kappa d\kappa,$$

which can be expressed in terms of the Weber-Schafheitlin integral with $\mu = \nu$.

We quote also the following result concerning the wave operators for H_μ , assuming now $\mu \in \mathbb{R}$:

Theorem 2.3 For $\mu, \nu > 1$, the Møller wave operators $\Omega_{\mu,\nu}^\pm$ associated to H_μ, H_ν exist and

$$\Omega_{\mu,\nu}^\pm := \lim_{t \rightarrow \pm\infty} e^{itH_\mu} e^{-itH_\nu} = e^{\pm i(\mu-\nu)\pi/2} \mathcal{F}_\mu \mathcal{F}_\nu.$$

In particular, the integral kernel of the operator $\Omega_{\mu,\nu}^\pm H_\nu^\gamma = H_\mu^\gamma \Omega_{\mu,\nu}^\pm$ may be useful in calculations. We have by the above considerations:

$$\Omega_{\mu,\nu}^\pm H_\nu^\gamma = e^{\pm i(\mu-\nu)\pi/2} \mathcal{F}_\mu Q^{2\gamma} \mathcal{F}_\nu$$

and its integral kernel is equal to

$$\Omega_{\mu,\nu}^\pm H_\nu^\gamma(x, y) = e^{\pm i(\mu-\nu)\pi/2} \int_0^\infty \kappa^{2\gamma} \sqrt{xy} J_\mu(x\kappa) J_\nu(y\kappa) \kappa d\kappa = e^{\pm i(\mu-\nu)\pi/2} \sqrt{\frac{x}{y}} y^{-2\gamma-1} \int_0^\infty \kappa^{2\gamma+1} J_\mu\left(\frac{x}{y}\kappa\right) J_\nu(\kappa) d\kappa,$$

where the last integral is of Weber-Schafheitlin type with exponent $2\gamma + 1$ and argument $\frac{x}{y}$.

3 The Weber-Schafheitlin integral with $\operatorname{Re} \rho < 1$

Proceeding as in [DF], we quote the following classic result [W] for the integral involving the modified Bessel function of the first and second kind, I_μ and K_μ :

Lemma 3.1 For $\operatorname{Re} z > 0$, $|z| > 1$, $\operatorname{Re}(\nu + \rho + 1) > |\operatorname{Re} \mu|$, one has

$$\int_0^\infty \kappa^\rho K_\mu(z\kappa) I_\nu(\kappa) d\kappa = \Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\rho-\mu+\nu}{2}\right) 2^{\rho-1} z^{-1-\rho-\nu} {}_2F_1\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \nu+1; z^{-2}\right).$$

Proof. The conditions for μ, ν, ρ , for the convergence of the integral, are established using asymptotic series for the Bessel functions of the corresponding type.

For the derivation of the integral, we first rescale the variable κ by the factor z^{-1} , expand $I_\nu(\kappa/z)$ as a power series and then use, assuming $|z| > 1$,

$$\int_0^\infty K_\mu(\kappa) \kappa^{\beta-1} d\kappa = \int_0^\infty \int_0^\infty e^{-u\kappa} (u^2 - 1)^{\mu-\frac{1}{2}} \kappa^{\beta-1} du d\kappa = 2^{\beta-2} \Gamma\left(\frac{\beta-\mu}{2}\right) \Gamma\left(\frac{\beta+\mu}{2}\right),$$

where we have substituted for K_μ the corresponding integral representation, and computed the obtained expression, integrating first with respect to κ . It remains to compare the obtained series with the hypergeometric ${}_2F_1$ function on the RHS. \square

We recall the relation between the Bessel function of the first kind J_μ and I_ν :

$$I_\nu(z) = i^{-\nu} J_\nu(iz).$$

Using

$$\begin{aligned} \int_0^\infty \kappa^\rho K_\mu(z\kappa) I_\nu(\kappa) d\kappa &= z^{-\rho-1} \int_0^\infty \kappa^\rho K_\mu(\kappa) I_\nu(\kappa/z) d\kappa, \\ \int_0^\infty \kappa^\rho K_\mu(z\kappa) J_\nu(\kappa) d\kappa &= z^{-\rho-1} \int_0^\infty \kappa^\rho K_\mu(\kappa) J_\nu(\kappa/z) d\kappa, \end{aligned}$$

we get as a straightforward corollary of Lemma 3.1:

Corollary 3.2 For $\operatorname{Re} z > 0$, $\operatorname{Re}(\nu + \rho + 1) > |\operatorname{Re} \mu|$,

$$\int_0^\infty \kappa^\rho K_\mu(z\kappa) J_\nu(\kappa) d\kappa = \Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\rho-\mu+\nu}{2}\right) 2^{\rho-1} z^{-1-\rho-\nu} {}_2F_1\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \nu+1; -z^{-2}\right). \quad (3.1)$$

Note that the superfluous assumption $|z| > 1$ has been eliminated by analytic continuation with respect to z , using the analyticity of the Gauss hypergeometric function ${}_2F_1$ on $\mathbb{C} \setminus [1, \infty[$.

We denote the Hankel function of the first and second kind, respectively — H^+ , H^- , and recall that they are related to K_μ as follows, for any $y \in \mathbb{C}$:

$$\begin{aligned} H_\mu^+(y) &= \frac{2}{i\pi} e^{\frac{-i\pi\mu}{2}} K_\mu(-iy), \\ H_\mu^-(y) &= -\frac{2}{i\pi} e^{\frac{i\pi\mu}{2}} K_\mu(iy). \end{aligned}$$

It follows that the integrals

$$\int_0^\infty \kappa^\rho H_\mu^\pm(x\kappa) J_\nu(\kappa) d\kappa$$

for $x \in \mathbb{R}_+$ are both the limiting case of $\int_0^\infty \kappa^\rho K_\mu(z\kappa) J_\nu(\kappa) d\kappa$ for purely imaginary z . We use the results obtained in Corollary 3.2 for $\operatorname{Re} z > 0$, setting first $z = \mp i(x \pm i\varepsilon)$, which gives:

$$\begin{aligned} \int_0^\infty \kappa^\rho H_\mu^\pm(x\kappa) J_\nu(\kappa) d\kappa &= \lim_{\varepsilon \searrow 0} \int_0^\infty \kappa^\rho H_\mu^\pm(x \pm i\varepsilon \kappa) J_\nu(\kappa) d\kappa \\ &= \frac{2}{i\pi} e^{\frac{\mp i\pi\mu}{2}} \lim_{\varepsilon \searrow 0} \int_0^\infty \kappa^\rho K_\mu(\mp i(x \pm i\varepsilon)\kappa) J_\nu(\kappa) d\kappa \\ &= \pm \frac{2^\rho}{i\pi} e^{\pm i\pi \frac{1+\rho-\mu+\nu}{2}} \Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\rho-\mu+\nu}{2}\right) \lim_{\varepsilon \searrow 0} f(x \pm i\varepsilon), \end{aligned}$$

where

$$f(x \pm i\varepsilon) := (x \pm i\varepsilon)^{-1-\rho-\nu} {}_2F_1^I\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \nu+1; (x \pm i\varepsilon)^{-2}\right).$$

Since we have also the relation $J_\mu = \frac{1}{2}(H_\mu^+ + H_\mu^-)$ [W], it follows that

$$\int_0^\infty \kappa^\rho J_\mu(x\kappa) J_\nu(\kappa) d\kappa = \frac{2^{\rho-1}}{i\pi} \Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\rho-\mu+\nu}{2}\right) \left[e^{i\pi \frac{1+\rho-\mu+\nu}{2}} \lim_{\varepsilon \searrow 0} f(x+i\varepsilon) - e^{-i\pi \frac{1+\rho-\mu+\nu}{2}} \lim_{\varepsilon \searrow 0} f(x-i\varepsilon) \right]. \quad (3.2)$$

Therefore, in order to derive the Weber-Schafheitlin integral, as well as the integrals involving H_μ^\pm , it is enough to examine the limit $\lim_{\varepsilon \searrow 0} f(x \pm i\varepsilon)$. Note that $f(z)$ depends on the parameters μ, ν, ρ and it will follow that it is convenient to treat the cases $\operatorname{Re} \rho < 0$ and $\operatorname{Re} \rho > 0$ separately.

Proposition 3.3 *For any $\mu, \nu \in \mathbb{C}$ and $\operatorname{Re} \rho < 0$ satisfying $\operatorname{Re}(\rho + \nu + 1) > |\mu|$, and $x \in \mathbb{R}_+$, the integral $\int_0^\infty \kappa^\rho J_\mu(x\kappa) J_\nu(\kappa) d\kappa$ is equal to:*

$$2^\rho \frac{\Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right)}{\Gamma\left(\frac{1-\rho+\mu-\nu}{2}\right)} x^\mu {}_2F_1^I\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho+\mu-\nu}{2}; \mu+1; x^{-2}\right)$$

for $x < 1$, and

$$2^\rho \frac{\Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right)}{\Gamma\left(\frac{1-\rho+\mu-\nu}{2}\right)} x^{-1-\rho-\nu} {}_2F_1^I\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \nu+1; x^{-2}\right)$$

for $x > 1$.

Proof. Consider the factor ${}_2F_1^I\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \nu+1; (x \pm i\varepsilon)^{-2}\right)$, appearing in the definition of $f(x \pm i\varepsilon)$. Denoting $a := \frac{1+\rho+\mu+\nu}{2}$, $b := \frac{1+\rho-\mu+\nu}{2}$ and $c := \nu+1$, we check that $\operatorname{Re}(c-a-b) = -\operatorname{Re} \rho < 0$, which ensures the limit $\varepsilon \searrow 0$ exists.

For $x > 1$, the argument of the ${}_2F_1$ function has real part smaller than 1, thus by analyticity it follows that the limits with $+i\varepsilon$ and $-i\varepsilon$ coincide. Obtaining the desired expression is then just a matter of rewriting the phase factors in terms of Γ functions, using

$$\frac{1}{2i} \left(e^{i\pi \frac{1+\rho-\mu+\nu}{2}} - e^{-i\pi \frac{1+\rho-\mu+\nu}{2}} \right) = \sin\left(\pi \frac{1+\rho-\mu+\nu}{2}\right) = \frac{\pi}{\Gamma\left(\frac{1+\rho+\nu-\mu}{2}\right) \Gamma\left(\frac{1-\rho-\nu+\mu}{2}\right)}.$$

For $x < 1$, we can use the result above with μ and ν interchanged, thanks to

$$\int_0^\infty \kappa^\rho J_\mu(x\kappa) J_\nu(\kappa) d\kappa = x^{-\rho-1} \int_0^\infty \kappa^\rho J_\mu(\kappa) J_\nu(\kappa/x) d\kappa.$$

□

4 The case $\operatorname{Re} \rho > 0$

Assuming $\operatorname{Re} \rho > 0$, we examine the limit

$$\lim_{\varepsilon \searrow 0} f(x \pm i\varepsilon) = \lim_{\varepsilon \searrow 0} (x \pm i\varepsilon)^{-1-\rho-\nu} {}_2F_1^I\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \nu+1; (x \pm i\varepsilon)^{-2}\right).$$

Denoting $a := \frac{1+\rho+\mu+\nu}{2}$, $b := \frac{1+\rho-\mu+\nu}{2}$ and $c := \nu+1$, we check that $\operatorname{Re}(c-a-b) = -\operatorname{Re} \rho > 0$, which implies a singular behaviour of the ${}_2F_1^I$ factor at $x = 1$ in the limit $\varepsilon \rightarrow 0$. We therefore use the following symmetry of the ${}_2F_1^I$ function:

$$\begin{aligned} {}_2F_1^I(a, b; c; z^{-2}) &= (1 - z^{-2})^{c-a-b} {}_2F_1^I(c-a, c-b; c; z^{-2}) \\ &= (1 - z^{-2})^{-\rho} {}_2F_1^I\left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; \nu+1; z^{-2}\right), \end{aligned}$$

in order to isolate the whole singular factor $(1 - z^{-2})^{-\rho}$, the second term on the RHS being well defined for all $z \in \mathbb{C}$. Consequently,

$$\begin{aligned} f(x \pm i\varepsilon) &= (x \pm i\varepsilon)^{-1-\rho-\nu} (1 - (x \pm i\varepsilon)^{-2})^{-\rho} {}_2F_1^I\left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; \nu+1; (x \pm i\varepsilon)^{-2}\right) \\ &= (x \pm i\varepsilon)^{-1+\rho-\nu} (x+1 \pm i\varepsilon)^{-\rho} (x-1 \pm i\varepsilon)^{-\rho} {}_2F_1^I\left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; \nu+1; (x \pm i\varepsilon)^{-2}\right). \end{aligned}$$

Because of the singular part (i.e. $(x-1 \pm i\varepsilon)^{-\rho}$), the limit $\varepsilon \searrow 0$ requires a distributional approach. Let us concentrate now on the first factors of the above expression. We will need the following:

Definition 4.1 We define $(x-1 \pm i0)^\lambda := \lim_{\varepsilon \searrow 0} (x-1 \pm i\varepsilon)^\lambda$, as distributions on \mathbb{R}_+ .

Remark 4.2 The distributions $(x-1 \pm i0)^\lambda$ are well defined and can be represented in a more explicit way as follows (see [H] for properties of the analogously defined distributions $(x \pm i0)^\lambda$ on \mathbb{R}):

- for $\operatorname{Re} \lambda > -1$, we have the equality between locally integrable functions

$$(x-1 \pm i0)^\lambda = e^{\pm i\lambda\pi} (x-1)_-^\lambda + (x-1)_+^\lambda,$$

where:

$$(x-1)_-^\lambda := \begin{cases} |x-1|^\lambda, & x < 1 \\ 0, & x \geq 1 \end{cases},$$

$$(x-1)_+^\lambda := \begin{cases} 0, & x < 1 \\ (x-1)^\lambda, & x \geq 1 \end{cases};$$

- for $\operatorname{Re} \lambda > -k$ and $\lambda \notin \mathbb{Z}$, where $k \in \mathbb{N}$, we have the equality

$$(x-1 \pm i0)^\lambda = e^{\pm i\lambda\pi} (x-1)_-^\lambda + (x-1)_+^\lambda,$$

where $(x-1)_\pm^\lambda$ are now distributions defined by their action on an arbitrary test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ (or equivalently, as the analytic continuation of the distributions defined in the preceding case):

$$\langle (x-1)_-^\lambda, \phi \rangle := \int_0^1 (1-x)^{\lambda+k} \phi^{(k)}(x) / ((\lambda+1) \dots (\lambda+k)) dx,$$

$$\langle (x-1)_+^\lambda, \phi \rangle := (-1)^k \int_1^\infty (x-1)^{\lambda+k} \phi^{(k)}(x) / ((\lambda+1) \dots (\lambda+k)) dx.$$

- for $\lambda = -k$, where $k \in \mathbb{N}$, we have

$$(x-1 \pm i0)^{-k} := (x-1)_-^{-k} + (-1)^k (x-1)_+^{-k} \pm (-1)^k i\pi \frac{\delta^{(k-1)}(x-1)}{(k-1)!},$$

where

$$\begin{aligned} \langle (x-1)_-^{-k}, \phi \rangle &:= (-1)^{k-1} \int_0^1 \log(1-x) \frac{\phi^{(k)}(x)}{(k-1)!} dx + (-1)^{k-1} \frac{\phi^{(k-1)}(1) \left(\sum_{j=1}^{k-1} j^{-1} \right)}{(k-1)!}, \\ \langle (x-1)_+^{-k}, \phi \rangle &:= - \int_1^\infty \log(x-1) \frac{\phi^{(k)}(x)}{(k-1)!} dx + \frac{\phi^{(k-1)}(1) \left(\sum_{j=1}^{k-1} j^{-1} \right)}{(k-1)!}. \end{aligned}$$

Lemma 4.3 *Let $\lambda \in \mathbb{C}$ and let $g(z)$ be a function holomorphic in the neighborhood of the halfline \mathbb{R}_+ . Then*

$$\lim_{\varepsilon \searrow 0} [g(x+i\varepsilon)(x-1 \pm i\varepsilon)^\lambda] = g(x)(x-1 \pm i0)^\lambda,$$

in the sense of distributions on \mathbb{R}_+ .

Proof. It is clear that

$$\lim_{\varepsilon \searrow 0} [g(x+i\varepsilon)(x-1 \pm i\varepsilon)^\lambda] = g(x)(x-1 \pm i0)^\lambda, \quad (4.1)$$

for $\operatorname{Re} \lambda > -1$, since the RHS is then a regular distribution and the convergence of each of the factors as functions is uniform on every compact subset of \mathbb{R}_+ . Assuming $\lambda \neq 0$, we differentiate (4.1) and obtain

$$\lim_{\varepsilon \searrow 0} [g'(x+i\varepsilon)(x-1 \pm i\varepsilon)^\lambda + \lambda g(x+i\varepsilon)(x-1 \pm i\varepsilon)^{\lambda-1}] = g'(x)(x-1 \pm i0)^\lambda + \lambda g(x)(x-1 \pm i0)^{\lambda-1},$$

where we have used the fact that $\frac{d}{dx}(x-1 \pm i0)^\lambda = \lambda(x-1 \pm i0)^{\lambda-1}$ [H]. We then use (4.1) again to subtract the first term of both sides. We have thus proved (4.1) for $\operatorname{Re} \lambda > -2, \lambda \neq -1$.

We consider the case $\lambda = -1$ separately, and differentiate instead the equality between the locally integrable functions:

$$\lim_{\varepsilon \searrow 0} [g(x+i\varepsilon) \log(x-1 \pm i\varepsilon)] = g(x) \log(x-1 \pm i0),$$

where $\log(x-1 \pm i0) := \lim_{\varepsilon \searrow 0} \log(x-1 \pm i\varepsilon) = \log|x-1| \pm i\pi\theta(x-1)$, and its distributional derivative is $(x-1 \pm i0)^{-1}$ (this can be seen by setting first $g(x) \equiv 1$ in the above equality and differentiating).

By induction, we prove (4.1) for the remaining values of $\lambda \in \mathbb{C}$. □

Recalling our expression for $f(x+i\varepsilon)$, we have

$$f(x+i\varepsilon) = (x+i\varepsilon)^{-1+\rho-\nu} (1+x+i\varepsilon)^{-\rho} (x-1+i\varepsilon)^{-\rho} q(x+i\varepsilon),$$

where

$$q(x+i\varepsilon) := {}_2F_1^I \left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; \nu+1; (x+i\varepsilon)^{-2} \right),$$

which converges uniformly on every compact subset of \mathbb{R}_+ to

$$q(x+i0) := {}_2F_1^I \left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; \nu+1; x^{-2} \right),$$

considered as a single-valued function. Note that the Gauss hypergeometric function ${}_2F_1$ has a branch cut $[1, \infty[$. Here, the limit is taken by approaching the halfline from below ($\operatorname{Im}(x+i\varepsilon)^{-2} < 0$), and we have denoted ${}_2F_1(a, b; c; x) = \lim_{\varepsilon \searrow 0} {}_2F_1(a, b; c; x-i\varepsilon)$ on the branch cut. On the other hand, the limit $\lim_{\varepsilon \searrow 0} f(x-i\varepsilon)$ corresponds to approaching the halfline from above in the argument of the ${}_2F_1$ factor, therefore

$$q(x-i0) := \lim_{\varepsilon \searrow 0} q(x-i\varepsilon)$$

is not equal to $q(x+i0)$.

Remark 4.4 The function $x \mapsto q(x \pm i0)$ is not differentiable, and therefore the meaning of the product $(x - 1 \pm i0)^{-\rho} q(x \pm i0)$ is unclear. To prove that it exists despite this apparent problem, we show that $q(x \pm i0)$ can be written as

$$q(x \pm i0) = h_1(x) + (x - 1 \pm i0)^\rho h_2^\pm(x), \quad (4.2)$$

where $h_1(x)$ is smooth and both $h_2^+(x)$, $h_2^-(x)$ belong to $L_1^{\text{loc}}(\mathbb{R}_+)$. Then, the equality

$$(x - 1 \pm i0)^{-\rho} q(x \pm i0) := (x - 1 \pm i0)^{-\rho} h_1(x) + h_2^\pm(x)$$

defines the desired product well, being the sum of a distribution multiplied by a smooth function and of a locally integrable function.

Proof. For $\rho \notin \mathbb{Z}$, the decomposition (4.2) is possible due to the following formula, holding for $z \in \mathbb{C}$ ([BS], eq. (B.9)):

$$\begin{aligned} {}_2F_1^I\left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; \nu+1; z\right) &= \frac{\pi}{\sin \pi \rho} \left\{ \frac{1}{\Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\rho-\mu+\nu}{2}\right)} {}_2F_1^I\left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; -\rho+1; 1-z\right) \right. \\ &\quad \left. - \frac{1}{\Gamma\left(\frac{1-\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\rho-\mu+\nu}{2}\right)} (1-z)^\rho {}_2F_1^I\left(\frac{1+\rho+\mu+\nu}{2}, \frac{1+\rho-\mu+\nu}{2}; \rho+1; 1-z\right) \right\}. \end{aligned} \quad (4.3)$$

We take $z = (x \pm i\varepsilon)^{-2}$ and pass to the limit $\varepsilon \searrow 0$ with both the expressions. Note that the only difference between the limit with $+i\varepsilon$ and $-i\varepsilon$ appears in the factor $\lim_{\varepsilon \searrow 0} (1 - (x \pm i\varepsilon)^{-2})^\rho = x^{-2\rho} (1+x)^\rho (x-1 \pm i0)^\rho$, since the ${}_2F_1$ factors on the RHS are analytic on the required domain. It is clear that (4.2) holds, with $h_2^+(x) = h_2^-(x)$ in particular.

For $\rho \in \mathbb{Z}$, we have instead ([BS], eq. (B.10)):

$$\begin{aligned} &{}_2F_1^I\left(\frac{1-\rho+\mu+\nu}{2}, \frac{1-\rho-\mu+\nu}{2}; \nu+1; z\right) = \\ &\frac{1}{\Gamma\left(\frac{1+\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\rho-\mu+\nu}{2}\right)} \sum_{k=0}^{\rho-1} \frac{(-1)^k (\rho-k-1)! \left(\frac{1-\rho+\mu+\nu}{2}\right)_k \left(\frac{1-\rho-\mu+\nu}{2}\right)_k (1-z)^k}{k!} + \\ &\frac{(-1)^\rho}{\Gamma\left(\frac{1-\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\rho-\mu+\nu}{2}\right)} (1-z)^\rho \sum_{k=0}^{\infty} \frac{\left(\frac{1+\rho+\mu+\nu}{2}\right)_k \left(\frac{1+\rho-\mu+\nu}{2}\right)_k}{(\rho+k)! k!} \{\psi(k+1) + \psi(\rho+k+1) \\ &\quad - \psi\left(\frac{1+\rho+\mu+\nu}{2} + k\right) - \psi\left(\frac{1+\rho-\mu+\nu}{2} + k\right) - \log(1-z)\} (1-z)^k, \end{aligned}$$

for $\frac{1-\rho \pm \mu + \nu}{2} \notin \mathbb{Z}$, where $\psi(y) = \Gamma'(y)/\Gamma(y)$.

As in the preceding case, we take $z = (x \pm i\varepsilon)^{-2}$ and pass to the limit. We obtain a similar decomposition. Note, however, that the limits $\log(1 - x \mp i0)$ differ, and thus $h_2^+(x) \neq h_2^-(x)$.

The cases where at least one of the parameters $\frac{1-\rho \pm \mu + \nu}{2}$ is an integer are treated the same way, only the functions $h_1(x)$, $h_2^\pm(x)$ being then different ([BS], eq. (B.11), (B.12)). \square

Proposition 4.5 In the sense of distributions on \mathbb{R}_+ ,

$$\lim_{\varepsilon \searrow 0} f(x \pm i\varepsilon) = x^{-1+\rho-\nu} (x+1)^{-\rho} \left[(x-1 \pm i0)^{-\rho} h_1(x) + h_2^\pm(x) \right],$$

where the functions $h_1(x)$ and $h_2^\pm(x)$ are defined in Remark 4.4.

Proof. We have to prove that as $\varepsilon \searrow 0$,

$$(x-1 \pm i\varepsilon)^{-\rho} (x \pm i\varepsilon)^{-1+\rho-\nu} (x+1 \pm i\varepsilon)^{-\rho} h_1(x \pm i\varepsilon) \longrightarrow (x-1 \pm i0)^{-\rho} x^{-1+\rho-\nu} (x+1)^{-\rho} h_1(x)$$

and

$$(x \pm i\varepsilon)^{-1+\rho-\nu}(x+1 \pm i\varepsilon)^{-\rho}(x-1 \pm i\varepsilon)^{-\rho}(x-1 \pm i\varepsilon)^{\rho}h_2(x \pm i\varepsilon) \longrightarrow x^{-1+\rho-\nu}(1+x)^{-\rho}h_2^{\pm}(x).$$

The first limit is a consequence of Lemma 4.3. The second one is clearly true, since the convergence of the corresponding functions is uniform on each compact subset of \mathbb{R}_+ . \square

Remark 4.6 *The case when $\rho \in \mathbb{Z}$ and at least one of the numbers $\frac{1-\rho \pm \mu + \nu}{2}$ is an integer are treated similarly. One can deduce from the expansions given in [BS] (eq. (B.11), (B.12)), the explicit expressions for the functions $h_1(x)$ and $h_2^{\pm}(x)$, following step by step the proof of Remark 4.4 in the degenerate case.*

Corollary 4.7 *Using Equation 3.2 and Proposition 4.5, we get the results gathered in Proposition 1.1. The formulae in Proposition 1.1 are written in terms of the distributions $(x-1)_{\pm}^{-\rho}$ rather than $(x-1 \pm i0)^{-\rho}$, using the relations listed in Remark 4.2.*

We end up commenting on some special cases, involving much simpler expressions.

Remark 4.8 *An explicit formula in the special case $\rho = 1$ has been derived in [KR]. It can be recovered from our general expression, by substituting the (well-known) equality*

$$(x-1 \pm i0)^{-1} = \text{Pv} \left(\frac{1}{x-1} \right) \mp i\pi\delta(x-1),$$

where Pv denotes the Cauchy principal value. Furthermore, $S_{\mu,\nu,1}(x) \equiv 1$ by definition, and it remains to use (4.3) back again to get ${}_2F_1$ functions with argument $x^{\pm 2}$ instead of $1 - x^{\pm 2}$.

Remark 4.9 *Much simpler formulae can be derived in the special case $1 - \rho + \mu \pm \nu = 0$, since the Gauss hypergeometric function with a parameter set to zero is trivial, i.e. ${}_2F_1(0, \dots; \cdot) \equiv 1$.*

Note that the above remarks hold for the integral involving H_{μ}^{\pm} instead of J_{μ} as well.

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MICHAŁ WROCHNA, *Research Training Group “Mathematical Structures in Modern Quantum Physics”*,
Mathematisches Institut, Universität Göttingen, Bunsenstr. 3-5, D - 37073 Göttingen, Germany
e-mail: wrochna@uni-math.gwdg.de